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# Hamiltonian Approach to Lagrangian Gauge Symmetries

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## Abstract

We reconsider the problem of finding all local symmetries of a Lagrangian. Our approach is completely Hamiltonian without any reference to the associated action. We present a simple algorithm for obtaining the restrictions on the gauge parameters entering in the definition of the generator of gauge transformations.

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The unravelling of gauge symmetries of a given action is an important problem which has received much attention in the past. Two main approaches have been followed in the literature: i) the Hamiltonian approach based on Dirac's conjecture [1, 2], where a suitable combination of the first class constraints is shown to be a generator of local symmetries of the Lagrangian, and ii) a purely Lagrangian approach, based on techniques used for discussing differential equations which are unsolvable with respect to the highest derivatives [3, 4, 5, 6].

With regard to the Hamiltonian approach essentially two different procedures have been followed recently: i) a hybrid approach where one departs from the requirement of the off-shell invariance of the *extended* action <sup>4</sup> under the local symmetry transformations generated by the phase space constraints, and then imposes a gauge condition whereby all Lagrange multipliers associated with secondary first-class constraints vanish [7]; ii) a purely algebraic approach based on the Poisson algebra of gauge generators with the constraints and the canonical Hamiltonian [8, 9]. In this case the restrictions on the gauge parameters have been obtained only for a special class of constrained systems.

In this paper we present a simple algorithm, based entirely on the *total* Hamiltonian approach, for obtaining the generator of the most general symmetry transformation of a given action, without ever making an explicit reference to the action itself. In order to simplify the discussion, and also for the sake of comparison, we restrict ourselves in the following to Hamiltonian systems with only irreducible first class constraints. The extension to systems with mixed first and second class constraints then involves a trivial step which we shall comment on at the end of this paper.

Consider a Hamiltonian system of  $2n$  degrees of freedom  $q_i, p_i, i = 1 \dots n$ ,

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<sup>4</sup>By extended action we mean the action constructed in terms of the extended Hamiltonian, in Dirac's terminology.

described by a canonical Hamiltonian  $H_c$  and a complete and irreducible set of (first class) primary constraints  $\Phi_{a_1} \approx 0$  ( $a_1 = 1, \dots, r$ ), and secondary constraints  $\Phi_{a_2} \approx 0$  ( $a_2 = r+1, \dots, N$ ), where  $r$  is the rank of the Hessian associated with the Lagrangian in question. We collect these constraints into a single vector with components  $\Phi_a$ ,  $a = 1, \dots, N$ . Following Dirac's conjecture, we make the following ansatz for the generator of gauge transformation:

$$G = \sum_{a=1}^N \epsilon^a(t, p, q, v) \Phi_a \quad (\text{late1})$$

where, as we shall see, it is in general necessary to allow the gauge parameters  $\epsilon_a$  to depend not only explicitly on time, but also on all phase space variables, including the Lagrange multipliers  $\{v^{a_1}\}$  (and their time derivatives) entering in the total Hamiltonian,

$$H_T = H_c + \sum_{a_1} v^{a_1} \Phi_{a_1} . \quad (\text{late2})$$

Here  $H_c$  is the canonical Hamiltonian, and  $\{\Phi_{a_1} \approx 0\}$  are the (first class) primary constraints.

An infinitesimal gauge transformation of a phase-space function  $F(q, p)$  is then given by

$$\delta F = [F, \Phi^a] \epsilon_a \quad (\text{late3})$$

where a summation over repeated indices is henceforth implied. Note that the gauge parameters appear outside the Poisson bracket. In principle we could have included the gauge parameters inside the Poisson bracket. These different definitions are weakly equivalent. As a result of the algebra

$$[H_C, \Phi_a] = V_a^b \Phi_b \quad (\text{late4})$$

$$[\Phi_a, \Phi_b] = C_{ab}^c \Phi_c \quad (\text{late5})$$

this weak equivalence continues to be true for the Poisson brackets of  $\delta F$  with either the canonical Hamiltonian  $H_C$  or the generator  $G$ . Since our analysis only involves this algebra, it is inconsequential whether the gauge parameters

are kept inside or outside the Poisson bracket. The structure functions  $C_{ab}^c$  and  $V_a^b$  can in general depend on the phase space variables.

We now notice that the action principle which leads to the Euler-Lagrange equations of motion, requires the commutativity of a general  $\delta$  variation with the time-differentiation. This commutativity need not hold for an arbitrary variation within the hamiltonian framework. Since our motivation is to abstract the symmetries of the action, we impose

$$\delta \frac{d}{dt} q_i = \frac{d}{dt} \delta q_i \quad ; i = 1, \dots, n. \quad (\text{late6})$$

as a fundamental requirement. This, as we now show, turns out to imply a non trivial condition on the gauge parameters and Lagrange multipliers.

As shown by Dirac [1], the Euler-Lagrange equations follow from the action principle  $\delta S_T = 0$ , where  $S_T$  is defined by

$$S_T = \int dt [p_i \dot{q}_i - H_T] \quad (\text{late7})$$

Moreover, the symmetries of the total action  $S_T$  are also the symmetries of  $S = \int dt L(q, \dot{q})$ , once the Hamilton equation of motion for  $\dot{q}_i$ , defining the relation between  $\dot{q}_i$ , and the momenta  $p_i$  as well as Lagrange multipliers, are used to eliminate the momenta and Lagrange multipliers in favour of all the velocities, including the undetermined ones. Since we are interested in the local symmetries of this action, we shall thus work with the total Hamiltonian.

The equations of motion within the hamiltonian framework, are given by,

$$\dot{q}_i = [q_i, H_T] = [q_i, H_c] + v^{a_1} [q_i, \Phi_{a_1}] \quad (\text{late8})$$

and,

$$\dot{p}_i = [p_i, H_T] = [p_i, H_c] + v^{a_1} [p_i, \Phi_{a_1}]$$

together with the constraint equations

$$\Phi_{a_1} = 0 \quad (\text{late9})$$

From (late8) and (late3) we obtain for the left hand side of (late6)

$$\delta \dot{q}_i = [[q_i, H_c], \Phi_a] \epsilon^a + v^{a_1} [[q_i, \Phi_{a_1}], \Phi_b] \epsilon^b + \delta v^{a_1} [q_i, \Phi_{a_1}] \quad (\text{late10})$$

and for the right hand side of the same equation,

$$\frac{d}{dt} \delta q_i = [[q_i, \Phi_a], H_c] \epsilon^a + v^{a_1} \epsilon^a [[q_i, \Phi_a], \Phi_{a_1}] + [q_i, \Phi_a] \frac{d\epsilon^a}{dt} . \quad (\text{late11})$$

Equating both expressions and making use of the Jacobi identity, we obtain

$$[[H_c, \Phi_a], q_i] \epsilon^a + v^{a_1} [[\Phi_{a_1}, \Phi_a], q_i] \epsilon^a - \delta v^{a_1} [q_i, \Phi_{a_1}] + [q_i, \Phi_a] \frac{d\epsilon^a}{dt} = 0 . \quad (\text{late12})$$

Using (late4) and (late5), we see that, on the constraint surface  $\{\Phi_a = 0\}$ , the above equation implies,

$$\left[ \frac{d\epsilon^b}{dt} - \epsilon^a [V_a^b + v^{a_1} C_{a_1 a}^b] \right] \frac{\partial \Phi_b}{\partial p_i} - \delta v^{a_1} \frac{\partial \Phi_{a_1}}{\partial p_i} = 0 . \quad (\text{late13})$$

Now, the first class nature and linear independence (irreducibility) of the constraints guarantees that each of these can be identified as a momentum conjugate to some coordinate, the precise mapping being effected by a canonical transformation. Since (late13) holds for all  $i$  one is led to the conditions

$$\delta v^{b_1} = \frac{d\epsilon^{b_1}}{dt} - \epsilon^a [V_a^{b_1} + v^{a_1} C_{a_1 a}^{b_1}] \quad (\text{late14})$$

$$0 = \frac{d\epsilon^{b_2}}{dt} - \epsilon^a [V_a^{b_2} + v^{a_1} C_{a_1 a}^{b_2}] \quad (\text{late15})$$

Note that in the above equations,  $\frac{d\epsilon^a}{dt}$  denotes the total time derivative as given by

$$\frac{d\epsilon^a}{dt} = \frac{D\epsilon^a}{Dt} + [\epsilon^a, H_T] \quad (\text{late16})$$

where, following the notation of Ref. [7],

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{v}^{a_1} \frac{\partial}{\partial v^{a_1}} + \ddot{v}^{a_1} \frac{\partial}{\partial \dot{v}^{a_1}} + \dots \quad (\text{late17})$$

The restrictions on the gauge parameters and the Lagrange multipliers found here are seen to agree with that of Ref.[7], obtained by looking at the

invariance of the total action considered as the gauge-fixed version of the extended action, defined in terms of the extended Hamiltonian

$$H_E = H_c + \sum \xi^a \Phi_a , \quad (\text{late18})$$

where the Lagrange multipliers  $\{\xi^{a2}\}$  are required to vanish by imposing suitable gauge conditions. Our analysis is also equally applicable to a dynamics determined by the extended Hamiltonian. The algebra now involves the full set of (primary and secondary) first class constraints, so that no condition emerges for the gauge parameters while the variation of the Lagrange multipliers  $\xi^a$  is given by

$$\delta \xi^a = \frac{d\epsilon^a}{dt} - \epsilon^b [V_b^a + \xi^c C_{cb}^a] . \quad (\text{late19})$$

Hence one is free to choose the gauge parameters  $\epsilon^a$  to be functions of time only. These equations again agree with those given in reference [7], as obtained by requiring the invariance of the corresponding extended action.

Let us conclude with some comments: Our requirement (late6) only involved the relation between the "velocities" and the canonical momenta and the arbitrary Lagrange multipliers. We have thus only used the "first" of Hamilton's equations, i.e., (late8)

Contrary to other procedures, our derivation was carried out on a purely (total) Hamiltonian level. As such we could have equally well worked with gauge transformations in the form,

$$\bar{\delta} F = [F, G] = [F, \epsilon^a \Phi_a] \quad (\text{late20})$$

because of the first-class nature of  $H_c$  and  $G$ . However, on the Lagrangian level, the two ways, (late3) and (late20) of writing the gauge transformation matters, since it is to be a symmetry also away from the constrained surface. As one easily checks, by explicitly looking at the off-shell invariance of the action (late7), it is the definition, as given by (late3), which leads to

the transformation law (late14) for the Lagrange multipliers, and conditions (late15) on the gauge parameters.

Finally we emphasize that the same discussion applies to the case where also second class constraints are present, with the simple replacement of  $H_c$  by the first class operator  $H^{(1)}$ , defined in the standard way by adding the contribution of the second class constraints whose Lagrange multipliers are now completely fixed.

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